Received: *April 29, 1998*, Accepted: *July 21, 1998*

Three-dimensional simplicial gravity and combinatorics of group presentations

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Abstract: We demonstrate how some problems arising in simplicial quantum gravity can be successfully addressed within the framework of combinatorial group theory. In particular, we argue that the number of simplicial 3-manifolds having a fixed homology type grows exponentially with the number of tetrahedra they are made of. We propose a model of 3D gravity interacting with scalar fermions, some restriction of which gives the 2-dimensional self-avoiding-loop-gas matrix model. We propose a qualitative picture of the phase structure of 3D simplicial gravity compatible with the numerical experiments and available analytical results.

Keywords: Lattice Models of Gravity, Matrix Models.

JHEP07(1998)010

Contents

1. Introduction

The success of the matrix models as a theory of 2D quantum gravity and non-critical strings has brought about a hope that analogous discrete approaches might be instructive in higher dimensions as well. The most natural way to introduce discretized quantum gravity is to consider simplicial complexes instead of continuous manifolds. Then, a path integral over metrics (which is the crux of any approach to quantum gravity) can be simply defined as a sum over all complexes having some fixed properties. For example, it is natural to restrict their topology. In the present paper, we consider only the 3-dimensional case, where, by definition, a simplicial complex is a collection of tetrahedra glued along their faces in such a way that any two of them can have at most one triangle in common.

To introduce metric properties, one can assume that complexes represent piece-wise linear manifolds glued of equilateral tetrahedra [1]. The volume is proportional to the number of them. In the continuum limit, as this number grows, the edge length, a, simultaneously tends to zero: $a \to 0$. In three dimensions, there is a one-to-one correspondence between classes of piece-wise linear and smooth manifolds. In other words, any homeomorphism can equally be approximated by either a piece-wise linear or smooth map. This property makes the model self-consistent.

Within a piece-wise linear approximation, the curvature is singular: the space is flat everywhere off edges. Therefore, one should consider integrated quantities, *e.g.*, the mean scalar curvature

$$\int d^3x \, \sqrt{g}R = a\Big(2\pi N_1 - 6N_3 \arccos\frac{1}{3}\Big)\,, \tag{1.1}$$

where a is the lattice spacing and $N_0,\ N_1,\ N_2,\ N_3$ are the numbers of vertices, edges, triangles and tetrahedra, respectively.

For manifolds in 3 dimensions, the Euler character vanishes $\chi = N_0 - N_1 + N_2 - N_3 = 0$. Together with the other constraint $N_2 = 2N_3$, it means that only 2 of the N_i's are independent, $N_1 = N_0 + N_3$, and a natural lattice action depends on 2 dimensionless parameters. Now, we are in a position to define the simplicial-gravity partition function [2, 3, 4, 5]:

$$\mathbf{Z}(\alpha, \mu) = \sum_{\{C\}} e^{\alpha N_0 - \mu N_3} = \sum_{N_3} Z_{N_3}(\alpha) e^{-\mu N_3}\,, \tag{1.2}$$

where $\sum_{\{C\}}$ is a sum over some class of complexes (*e.g.,* of a fixed topology).

It can be shown that, if N_3 is fixed, N_0 is restricted from above as $N_0 < \frac{1}{3}N_3 + const$. Therefore, it is natural, keeping α fixed, to tend μ to its critical value, μ_c, at which the sum over N_3 in Eq. (1.2) becomes divergent. In the vicinity of μ_c one could expect to find critical behavior corresponding to a continuum limit of the model (of course, it can require a fine-tuning of the second parameter α).

For the model to be physically consistent, the free energy per unit volume $\frac{1}{N_3} \cdot \log Z_{N_3}(\alpha)$ has to be finite, which means that the series over N_3 in Eq. (1.2) has to have a finite radius of convergence: $0 < \mu_c < +\infty$. If it holds, statistical models defined on such an ensemble of fluctuating lattices possess reasonable thermodynamical behavior. Thus one can hope to describe matter coupled to quantum gravity. This program has been successfully carried out in two dimensions with help of the matrix model technique [7, 8]. In the 3D case, analytic tools are lacking and numerical simulations have so far been the only source of information [3, 4].

Many questions, which are simple in two dimensions, become stubborn in three. For example, 2-manifolds are classified according to values of the Euler character, therefore topology, homotopy and homology are actually equivalent descriptions. In three dimensions, sets of manifolds characterized according to these 3 criteria are essentially different. A natural question to ask is how large a class of complexes $\{C\}$ in Eq. (1.2) can be. We shall argue that it is sufficient to fixe only the first homology group $H_1(C, Z)$ in order to have a finite value of μ_c. It can be reformulated as the following statement: "The number of simplicial manifolds constructed of the given number of tetrahedra, N_3, and having a fixed homology type (*viz.,* the first Betti number and the torsion coefficients) grows at most exponentially as a function of N_3".

Let us notice that the existence of the exponential bound is not broken down, if one weights every complex with a positive weight growing at most exponentially with N_3. As well, if one manages to represent a sum over complexes as a weighted sum over another class of objects and to prove that the weights are exponentially bounded, then one can simply estimate the number of objects in this new class ignoring the weights.

The simplest example of a weighted sum of complexes is given by the canonical partition function $Z_{N_3}(\alpha)$ introduced in Eq. (1.2). Another example of physical interest is the partition function in the presence of free matter fields. To introduce it let us consider the set of polyhedral complexes dual to simplicial ones. Their 1-skeletons are some 4-valent graphs, $G^{(4)}$, whose vertices correspond to tetrahedra, and links are dual to triangles. Such a graph can be defined by the adjacency matrix

$$G^{(4)}_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected by a link} \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.3}$$

Of course, the set of 4-valent graphs is not identical to the totality of simplicial complexes. Similarly, in two dimensions, the set of ordinary ϕ^3 Feynmann diagrams is different from that of triangulations – one has to introduce the notion of fat graphs to establish the equivalence.

If one attaches the n-component vector x^μ_i ($\mu = 1, \ldots, n$) to the i-th vertex, the corresponding gaussian integral can be performed explicitly for every given graph $G^{(4)}$:

$$Z_{matter}(G^{(4)}) = \int \prod_{i=1}^{N_3-1} \prod_{\mu=1}^{n} dx^\mu_i \exp\Big[-\frac{1}{2}\sum_{i,j=1}^{N_3-1} G^{(4)}_{ij}(x^\nu_i - x^\nu_j)^2\Big] = \Big(\det{}' L^{(4)}\Big)^{-\frac{n}{2}}, \tag{1.4}$$

where the discrete Laplacian is given by

$$L^{(4)}_{ij} = 4\delta_{ij} - G^{(4)}_{ij}\,. \tag{1.5}$$

There is the nice combinatorial representation of the determinant given by Kirchhoff's theorem [9]:

$$\det{}' L = |T(G)|\,, \tag{1.6}$$

where $T(G)$ is the set of all possible spanning trees embedded into a graph G or, equivalently, a number of connected trees which can be obtained from the graph by cutting its links. To kill the zero mode in Eq. (1.4), we fixed the field at the N_3'th vertex. The theorem states that $\det{}' L$ does not depend on a choice of the vertex. Therefore, $|T(G)|$ is the number of rooted trees.

The number of spanning trees of a k-valent graph $G^{(k)}_n$ with n vertices can be estimated from above as [9]

$$|T(G^{(k)}_n)| \le \frac{1}{n}\left(\frac{nk}{n-1}\right)^{n-1}. \tag{1.7}$$

The case we are particularly interested in is the 2-component Grassmann field, where we obtain the gravity+matter partition function in the form

$$\mathbf{Z}^{(1)}(\alpha, \mu) = \sum_{\{C\}} \det{}' L\, e^{\alpha N_0 - \mu N_3} = \sum_{N_3} Z^{(1)}_{N_3}(\alpha) e^{-\mu N_3}\,. \tag{1.8}$$

The weights, $\det{}' L$, are positive integers, hence,

$$Z_{N_3}(\alpha) < Z^{(1)}_{N_3}(\alpha)\,. \tag{1.9}$$

In two dimensions, this type of matter has the central charge $c = -2$ and the corresponding matrix model has been solved explicitly by purely combinatorial means [7]. We can repeat the same trick in 3 dimensions:[1]

$$\mathbf{Z}^{(1)}(\alpha, \mu) = \sum_{\{C\}} \det{}' L \, e^{\alpha N_0 - \mu N_3} = \sum_{\{C\}} \sum_{\{\widetilde{T}(C)\}} e^{\alpha N_0 - \mu N_3} = \sum_{\{\widetilde{T}\}} \sum_{\{C(\widetilde{T})\}} e^{\alpha N_0 - \mu N_3} . \quad (1.10)$$

Namely, we have the sum over complexes, $\sum_{\{C\}}$, weighted with the number of spanning trees in the dual 1-skeleton of each complex, $\sum_{\{\widetilde{T}(C)\}}$. We can obtain exactly the same configurations by taking the sum over all possible trees, $\sum_{\{\widetilde{T}\}}$, weighted with the number of complexes, $\sum_{\{C(\widetilde{T})\}}$, which can be recovered from a given tree $\widetilde{T}$ by restoring cut links.

In three dimensions, the appearing trees correspond to spherical simplicial balls obtained starting with a single tetrahedron by subsequently gluing other tetrahedra to faces of the boundary. If we denote as n_0, n_1 and n_2 numbers of vertices, edges and triangles on the boundary of a ball, we find that

$$n_0 = N_3 + 3 \qquad n_2 = 2(N_3 + 1) . \quad (1.11)$$

In contrast to the 2-dimensional case, it is extremely difficult in three dimensions to control a topology of complexes constructed from a ball by pairwise identifications of triangles belonging to the boundary. Therefore, we shall not address the homeomorphic properties of complexes in this paper. On the other hand, homotopy information is nicely coded in the fundamental group, which is going to be the basic object for us. The combinatorial group theory provides us with the most adequate mathematical framework within which all physical questions can be asked and, hopefully, answered.

The outline of the paper is the following.
In Section 2 we collect some basic definitions and results from combinatorial group theory and 3-dimensional topology.
In Section 3 we give an interpretation of the 3D gravity partition function in terms of group presentations.
In Section 4 we argue that the number of simplicial complexes of a given volume having fixed homology type grows exponentially as a function of the volume.
In Section 5 we introduce a reduced model and establish a connection with the 2D loop gas matrix model.
Section 6 is devoted to a discussion.

The present paper can be regarded as an extended version of the previous one [6]. A reader must be aware that Ref. [6] contains some incorrect statements. In particular, the proof of the main theorem stated in it is not mathematically satisfactory.

[1]This model was first proposed by V.A.Kazakov as a natural generalization of the corresponding 2-dimensional matrix model.

2. Complexes and group presentations

Although almost all the mathematical material collected in this section can be found in the standard reference books on lower dimensional topology and combinatorial group theory (see for example [10, 11]), it is convenient to repeat it here for reader's conveninience. Let us start with some definitions.

If it is not specifically stated, by a complex we always mean a finite cell (polyhedral) decomposition of a compact closed orientable 3-manifold. Any complex, C, can be constructed by first assembling the 1-dimensional cells (giving the 1-skeleton, K_1), then attaching 2-cells (giving the 2-skeleton, K_2) and analogously for K_3.

A simplicial complex is a complex in which k-dimensional cells are k-simplexes, *i.e.,* points, segments, triangles and tetrahedra. The intersection of any two simplexes either is empty or consists of exactly one less dimensional simplex. Any simplex in a simplicial complex can be unambiguously determined, including an orientation, by a list of vertices from its boundary. It means, in particular, that any edge connects a pair of distinct vertices.

The fundamental group $\pi_1(C)$ is determined by the 2-skeleton of a complex: $\pi_1(C) \equiv \pi_1(K_2)$. The standard algorithm to read off a presentation $\pi_1(C)$ is the following:

1. Construct a spanning tree T of the 1-skeleton of C.

2. Put into correspondence a generator, a_i, to every link of K_1 which is not in T (*i.e.,* to each cut link). Links belonging to T are formally associated with the identity, 1.

3. Fixe some orientation of the links of K_1; a change of the orientation corresponds to the inversion of the generator: $a_i \to a_i^{-1}$.

4. For every face in the 2-skeleton, the cyclic order of the oriented links forming its perimeter gives a relation: $r_j = a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \dots a_{i_\ell}^{\epsilon_\ell} = 1$; $\epsilon_i = \pm 1$ according to the orientation of the i'th link (one puts the identity, 1, on places corresponding to the links which are in T).

As any spanning tree cuts all loops in K_1, all relations obtained in this way are non-trivial (the trivial relation is, by definition, of the form 1=1).

In this way one finds the fundamental group presentation

$$\pi_1(C) = \langle a_1 a_2 \dots a_n | r_1 r_2 \dots r_m \rangle\,, , \tag{2.1}$$

where m equals the number of 2-cells in C and n is not bigger that the number of 1-cells.

This procedure gives a nonreduced group presentation in the sense that presentations differing by the trivial cancellations, $aa^{-1} \to 1$ and $a^{-1}a \to 1$, correspond to distinct complexes. The simplest example is the following identical presentations of the trivial group

$$\begin{aligned} P'_E &= \langle a | a, a, a \rangle \\ P''_E &= \langle a | a a^{-1} a \rangle \,, \end{aligned} \tag{2.2}$$

In both cases K_2 consists of the single triangle but, in the second case, its edges are identified in such a way that the corresponding complex has only one 0-cell and only one 1-cell. The 2-skeleton here is the so-called Zeeman's "Dunce hat"[(2)], a closed contractible fake-surface. Both 3-complexes represent S^3 [12].

Therefore, in order to maintain correspondence between presentations and complexes, one has to consider nonreduced presentations. However, as we shall see later, in the case of simplicial 3-complexes, this subtlety can easily be overcome. Therefore, in what follows, we shall use the term "presentation" hoping it will always be clear from a context what is meant in every particular case.

Two presentations describe the same complex, if they can be transformed into each other by a sequence of the following transformations:

1. permutation of generators or relators;
2. inversion of a generator, $a_i \to a_i^{-1}$, or a relator, $r = a_1 a_2 \dots a_k \to r^{-1} = a_k^{-1} \dots a_2^{-1} a_1^{-1}$;
3. cyclic shift of a relator: $r = a_1 a_2 \dots a_k \to a_1^{-1} r a_1 = a_2 \dots a_k a_1$;

These moves obviously divide the totality of all presentations into equivalence classes. We shall always identify presentations from the same class.

It is known that all 3-manifolds possess finitely presentable fundamental groups. Therefore, any of them can be described as the factor group

$$G = F(X)/N \tag{2.3}$$

where $F(X)$ is the free group on n generators, $X \equiv (a_1 a_2 \dots a_n)$, and N is the normal closure in $F(X)$ of a subgroup generated by some finite set of words $R \subset F(X)$. The words constituting R are the relators and this is an equivalent description of the presentation. R form a basis of the subgroup $\langle R \rangle$. Any other set of generators of $\langle R \rangle$ can be obtained from R by applying the Nielsen transformations:

N1) replace r_j by r_j^{-1}.

N2) replace r_j by $r_j r_i$ with $i \neq j$.

This pair of transformations generate a group including permutations as a subgroup. Any sequence of the Nielsen transformations carries a presentation $\langle X | R \rangle$ into an equivalent one. Let us notice that they do not change the number of generators.

2) I am indebted to J.Howie for pointing me out this example.

To obtain the normal closure of $\langle R \rangle$, one has to add the third move, namely, the conjugation by an arbitrary word $w \in F(X)$:

N3) replace r_j by $w r_j w^{-1}$.

In order to establish the isomorphism of two groups, one has to use the Tietze transformations:

T1) add or remove a relator which is a consequence of other relators:

$$\langle X | R \rangle \longleftrightarrow \langle X | R, y \rangle$$

where $y \in N$, N is the normal closure of $\langle R \rangle$ in F(X).

T2) add or remove a generator which is a word in the others:

$$\langle X | R \rangle \longleftrightarrow \langle X, x | R, x^{-1} w \rangle$$

where R and w do not involve x; w is arbitrary. This move is called the prolongation.

It can be shown that these moves connect any two presentations of the same group. It should be said that there is no correspondence between the Tietze moves and the transformations of complexes: in most cases they yield presentations having no realization in the form of 3-complexes.

One of the most profound connections between homotopy theory and group presentations is established via the so-called formal defomations of complexes [13]. Let us introduce this notion. An elementary n-expansion, $C' \nearrow C''$, is an attachment to C' of an n-ball B_n along all of the boundary ∂B_n except one $(n-1)$-cell: $C'' = C' \bigcup B_n$, $C' \bigcap B_n \cong B_{n-1}$. If the complexes in question are simplicial, the ball is simply the single tetrahedron. An elementary n-collapse, $C' \searrow C''$, is the inverse of the elementary n-expansion. A formal n-deformation, $C' \overset{n}{\nearrow\searrow} C''$, is a finite sequence of the elementary expansions and collapses in which the maximum dimension of the balls equals n.

If a complex C can be n-deformed to a point, $*$, one says that C is n-collapsible. This term should not be confused with contractability, which stays for the simply-connected complexes, $\pi_1(C) = 1$.

As was proven by P.Wright [14], any two contractible 2-complexes, $\pi_1(K_2') = \pi_1(K_2'') = 1$, can be transformed into each other by a finite sequence of the formal deformations not exceeding the dimension 3, $K_2' \overset{3}{\nearrow\searrow} K_2''$, if and only if the corresponding presentations read off in the standard way from the two 2-complexes K_2' and K_2'' can be transformed into each other by a finite sequence of the Nielsen transformations extended by the trivial prolongation. We shall call this set of moves the AC transformations:

AC1) cancellation of aa^{-1} and its inverse.

AC2) $r_i \to r_i^{-1}$ and $r_i \to r_i r_j$ with $i \neq j$.

AC3) $r_i \to w r_i w^{-1}$ where $w \in F(X)$.

AC4) $\langle X|R\rangle \longleftrightarrow \langle X, a|R, a\rangle$ where $a \notin X$, $a \notin R$.

These transformations will play an important role in the sequel.

It is not known whether arbitrary two 2-complexes having the same fundamental group can always be connected by a chain of the formal 3-deformations, in other words, whether in three dimensions the simple homotopy type and the 3-deformation type coincide. The famous Andrews-Curtis conjecture reads: *for any contractible compact* K_2, $K_2 \overset{3}{\nearrow\!\searrow} *$ *holds.* It seems that this conjecture is commonly believed to be false [15]. In any case, as we have decided to forget about the notion of homeomorphy, the 3-deformation classes of 2-spines of simplicial 3-manifolds seem to be the most natural finest classification in our context.

Another theorem of P.Wright [14] says that $K_2' \overset{2}{\nearrow\!\searrow} K_2''$ holds if and only if the corresponding presentations can be transformed into each other by a sequence of arbitrary prolongations (T2). It is important that the cancellation of aa^{-1} is not allowed as an independent move. For example, Zeeman's "Dunce hat" is not 2-collapsible albeit contractible.

It is known that the first homology group is the abelianized fundamental group: $H_1(C, Z) = \pi_1(C)/[\pi_1(C), \pi_1(C)]$. In the abelian case, it is convenient to represent the defining relations in the form of a linear system of equations:

$$\widehat{P} = \langle x_1 \ldots x_n | \sum_{j=1}^{n} \widehat{R}_{ij} x_j \rangle \,, \tag{2.4}$$

where the relation matrix $\widehat{R}$ has integer entries.

Another set of free generators can be obtained by rotating the vector $(x_1 \ldots x_n)$ by an invertible unimodular integral matrix:

$$x_i' = \sum_{j=1}^{n} M_{ij} x_j \qquad M \in SL(n, Z) \tag{2.5}$$

and analogously for relators.

It is known that pre- and post-multiplications by $SL(n, Z)$ matrices can be expanded into sequences of the following elementary operations

1. permutation of rows.

2. multiplication of all elements in a row by -1.

3. addition of a row to another row.

and the same for columns.

These operations correspond to the Nielsen transformations and give a homomorphism of the group of the Nielsen moves into $SL(n,Z)$. The kernel is generated by the conjugations, $r_j \to r_i r_j r_i^{-1}$.

An arbitrary rectangular $n \times m$ matrix can be transformed into the diagonal form

$$D = diag(d_1, d_2, \ldots, d_k) \qquad k = \min(n,m) \tag{2.6}$$

where d_i are non-negative integers and d_i divides d_{i+1} $(d_i | d_{i+1})$.

Thus, any abelian group has the unique presentation of the form

$$\widehat{G} = \langle x_1 \ldots x_n | x_1^{d_1}, x_2^{d_2}, \ldots, x_q^{d_q}, C \rangle \cong Z_{p_1} \times Z_{p_2} \times \ldots Z_{p_q} \times \underbrace{Z \times \ldots \times Z}_{b_1 - times}, \tag{2.7}$$

where $C = \{x_i x_j x_i^{-1} x_j^{-1}\}$ is the commutator subgroup of $F(X)$; Z_p is the cyclic group $\langle x | x^p \rangle$; Z is the infinite cyclic group $\langle x | \ \rangle$; $p_1 > 1$; $p_q \neq 0$; $p_i | p_{i+1}$. The number of Z factors is called the first Betti number, b_1, and the integers $(p_1, p_2, \ldots, p_q)$ are the torsion coefficients.

If $\widehat{R}$ is a square matrix such that $\det \widehat{R} \neq 0$, then it determines a finite abelian group $\widehat{G}$ of the order $|\widehat{G}| = \pm \det \widehat{R}$.

It is well known that not all groups can be fundamental groups of 3-manifolds and not all presentations of a given group can be realized in the form of a 2-skeleton of some 3-complex. The obvious (but not the only one) source of the restrictions is the Poincaré duality. Given a 3-complex C, the dual complex $\widetilde{C}$ is constructed by putting k-dimensional cells of C into correspondence to $(3-k)$-cells of $\widetilde{C}$. A presentation of $\pi(\widetilde{C})$ can be read off from the 2-skeleton, $\widetilde{K}_2$, of $\widetilde{C}$. As we consider manifolds, ,$\pi_1(\widetilde{C}) \cong \pi_1(C)$. It implies a kind of duality between generators and relators in presentations of the fundamental groups of 3-manifolds.

Let us fixe a spanning tree $\widetilde{T}$ of the dual 1-skeleton, $\widetilde{K}_1$. Relators in Eq. (2.1) are in correspondence with links of $\widetilde{K}_1$. This set of them is obviously excessive. A minimal set is given by links which is not in $\widetilde{T} \in \widetilde{K}_1$. In this case, relators of $\pi_1(C)$ correspond to generators of $\pi_1(\widetilde{C})$ and *vice versa*. In particular, their numbers equal each other. Such presentations are called balanced.

3. Combinatorial meaning of the partition function

Now we are in a position to interpret the partition function $\mathbf{Z}^{(1)}(\alpha, \mu)$ introduced in Eq. (1.10) in terms of combinatorial group theory. Every simplicial ball from $\sum_{\{\widetilde{T}\}}$ determines a set of $N_3 + 1$ generators of $\pi_1(\widetilde{C})$. Any 3-manyfold M_3 can be obtained out of some ball by gluing its boundary triangles pairwise. This process induces corresponding identifications of edges and vertices. The glued triangles form a 2-dimensional spine[3], $K_2 \subset M_3$, having the Euler character $\chi(K_2) = 1$. Every edge in K_2 gives a

[3] By definition, a spine is a 2-skeleton in a cell decomposition with a single 3-cell.

relator of the fundamental group, $\pi_1(M_3)$, determined by a cyclic order of oriented triangles attached to it. Thus we find

$$\mathbf{Z}^{(1)}_{\pi_1}(\alpha,\mu) = \sum_{N_3,N_1}\sum_{\{P_{N_3+1,N_1}\}} e^{\alpha N_1-(\alpha+\mu)N_3}\,, \tag{3.1}$$

where $\sum_{\{P_{N_3+1,N_1}\}}$ is the sum over all nonreduced presentations on N_3+1 generators and N_1 relators associated with the 2-spines of simplicial complexes from a given class ($[\pi_1]=\{C:\ \pi_1(C)\text{ fixed}\}$):

$$P_{n,m}\equiv\langle X_n|R_m\rangle=\langle a_1a_2\dots a_n|r_1r_2\dots r_m\rangle\in[\pi_1]\,, \tag{3.2}$$

where r_i's are nonreduced words in $a_j^{\pm1}$. Every generator $a_i\in X_n$ appeares exactly 3 times in R_m, simply because the triangle has 3 edges.

Any spanning tree T of the 1-skeleton K_1 of the 2-spine K_2 ($T\subset K_1\subset K_2$) fixes a minimal subset of the relators $R'_{N_3+1}\subset R_{N_1}$. The deformation retract of a tree is a point: $T\searrow *$, and the fundamental group of any 3-manifold possesses a balanced presentation.

It is convenient to weight every complex with the number of spanning trees $T\subset K_1\subset K_2$. It can be done by introducing another system of free matter fields attached to vertices of triangulations. Then, analogously to Eq. (1.10), we define the partition function

$$\mathbf{Z}^{(2)}(\alpha,\mu)=\sum_{\{C\}}\det{}'L\ \det{}'\widetilde{L}\ e^{\alpha N_0-\mu N_3}=\sum_{N_3}e^{-\mu N_3}\sum_{\{H_{N_3}\in[\pi_1]\}}\boldsymbol{\Upsilon}(H_{N_3},\alpha)\,, \tag{3.3}$$

where $\sum_{\{H_{N_3}\in[\pi_1]\}}$ goes over all balanced presentations from a given homotopy class: $\{H_{N_3}\equiv\langle X_{N_3+1}|R'_{N_3+1}\rangle\}=[\pi_1]$. All such presentations can be enumerated. Then, there exists an algorithm [16] allowing, in principle, for determining if a corresponding 2-complex is a spine of some 3-manifold.

$$\boldsymbol{\Upsilon}(H_{N_3},\alpha)=\sum_{N_0}\sum_{\{P_{N_3+1,N_3+N_0}\cong H_{N_3}\}}e^{\alpha N_0}=\sum_{N_0}e^{\alpha N_0}\Upsilon_{N_0}(H_{N_3})\,, \tag{3.4}$$

where $\sum_{\{P_{N_3+1,N_3+N_0}\cong H_{N_3}\}}$ is the sum over all nonreduced presentations $\langle X_{N_3+1}|R_{N_3+N_0}\rangle$ which can be deduced from a given balanced presentation $H_{N_3}=\langle X_{N_3+1}|R_{N_3+1}\rangle$ following a pattern of a spanning tree in some simplicial complex. The last equality is a definition of $\Upsilon_{N_0}(H_{N_3})$.

In this way we divide the counting problem into two steps: we estimate first the number of the balanced presentations in a class $\{H_{N_3}\}$ and then the number of simplicial complexes giving any given balanced presentation from $\{H_{N_3}\}$. The last number, $\Upsilon_{N_0}(H_{N_3})$, is exponentially bounded because any configuration has the spanning tree structure and the bound is provided by the number of corresponding trees. To estimate the first sum, $|H_{N_3}\in\ [\pi_1]|$, is a fundamentally difficult combinatorial problem.

The balanced presentations enjoy the duality property. Therefore $\sum_{\{H_{N_3}\}}$ can be equivalently described as the sum over the balanced presentations in which the maximal length of relators does not exceed 3: $\ell(r_j) \leq 3,\ \forall j$. In this dual description, the cyllables $a_i a_i^{-1}$ and a_i^2 never appear in the relators, because in that case there would exist a circularly closed edge, which is forbidden by the definition of the simplicial complex. Therefore we can consider reduced relators without loss of generality.

4. Estimates for abelian presentations

Let us formulate the following simple fact as

PROPOSITION 1. If the maximum length of relators equals 3 $[\ell(r_j) \leq 3,\ \forall j]$, then there is at most 2^n balanced group presentations on n relators, $\langle X_n | R_n \rangle$, having the same abelianization.

PROOF. As $\ell(r_i) \leq 3$, commutators, $aba^{-1}b^{-1}$, can not appear. Given 3 letters a, b and c, there are only 2 inequivalent relators having the abelianization abc:

$$[abc] = [bca] = [cab] \qquad\qquad [acb] = [cba] = [bac] \quad \Box\,.$$

Let us consider the (non-trivial) case when $\pi_1(C)$ has the trivial abelianization: $H_1(C, Z) = 0$. Balanced presentations of the trivial abelian group are in correspondence with the unimodular integral matrices[4]

$$\widehat{P}_E = \langle x_1 \ldots x_n | \sum_{j=1}^{n} \widehat{R}_{ij} x_j \rangle \qquad .\widehat{R} \in SL(n, Z) \tag{4.1}$$

We are interested in the subset corresponding to triangulations:

$$\mathcal{M}_n = \{\widehat{R} \in SL(n, R) : \sum_{j=1}^{n} \widehat{R}_{ij}^2 \leq 3,\ \forall i\}\,. \tag{4.2}$$

For all matrices $\widehat{R} \in \mathcal{M}_n$, the allowed matrix elements are $\widehat{R}_{ij} = 0, \pm 1$, and there are at most 3 non-zero elements in each row. We can independently permute rows and columns and multiply them by -1. It means the double coset $O(n, Z) \backslash \mathcal{M}_n / O(n, Z)$. The group $O(n, Z) = S_n * Z_2$ is the free product of the symmetric group S_n and the group generated by the multiplication by the diagonal matrix $diag(-1, +1, \ldots, +1)$.

An $\widehat{R}$ matrix can be further reduced. If the i'th row has only one non-zero entry, $\widehat{R}_{ij} \neq 0$, then we can remove it along with the j'th column by applying the AC2 and AC4 moves. If there are two non-zero entries, $\widehat{R}_{ij} \neq 0$ and $\widehat{R}_{ik} \neq 0$, we nullify one of them by using the AC2 move on columns. After that we can repeat the previous step in order to decrease the size of the matrix. Therefore, without loss of generality, we assume that $(\widehat{R}\widehat{R}^{\mathrm{T}})_{jj} = 3$ $(j = 1, \ldots, n)$ and $(\widehat{R}^{\mathrm{T}}\widehat{R})_{jj} \geq 3$ $(j = 1, \ldots, n)$.

PROPOSITION 2. Let a simplicial complex C give (maybe after the described reduction) a presentation matrix such that $(\widehat{R}\widehat{R}^{\mathrm{T}})_{jj} = 3\,\forall j$. If C is a 3-manifold, then $\sum_{i=1}^{n} |\widehat{R}_{ij}| = 3,\ \forall j$.

[4] In principle, $\det \widehat{R} = \pm 1$. We choose $+1$ for convenience.

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PROOF. The reduced complex has only one 0-cell σ^0, n 1-cells σ^1_k and n 2-cells σ^2_k. Let us consider the regular neighborhood of its 1-skeleton K_1 (cf. Ref. [16]). The neigborhood of σ^0 is a 3-ball β^3. Its boundary is the 2-sphere $\partial\beta^3 \cong S^2$. The neigborhoods of 1-cells σ^1_k are solid tori. They intersect the sphere, $\partial\beta^3$, in a disjoint collection of disks. The 2-cells σ^2_k intersect $\partial\beta^3$ in a number of simple arcs connecting the disks pairwise. The number of disks equals $2n$, while that of arcs equals $3n$. Any such configuration defines a graph in a natural way. If any edge E of a graph connects distinct vertices and the valency of each vertex V is at least 3, then $2\#E \geq 3\#V$. As in the case in question $2\#E = 3\#V = 6n$, we conclude that the valency of each vertex equals 3. $\Box$

Let us denote this restricted set of matrices as $\widehat{\mathcal{M}}^3_n \equiv \{\widehat{R} \in SL(n,Z) \mid (\widehat{R}^{\mathrm{T}}\widehat{R})_{jj} = (\widehat{R}\widehat{R}^{\mathrm{T}})_{jj} = 3\ \forall j\}$. If $\widehat{R}$ has a block form, $\det \widehat{R}$ equals the product of determinants of the blocks. Let us consider the irreducible case. Any matrix $\widehat{R} \in \widehat{\mathcal{M}}^3_n$ can be transformed into the form $\widehat{R} = I - B$. Then B can be uniquely represented as the sum of two $O(n,Z)$ matrices $B = P_1 + P_2$: $P_{1,2} \in O(n,Z)$. This form has the residual symmetry under conjugate permutations: $\widehat{R} \to \sigma\widehat{R}\sigma^{-1}$, $\sigma \in O(n,Z)$. Therefore we can take the conjugacy classes of the B matrices as representatives for the double coset. For example, we can specify the cycle structure of $P_1^{-1}P_2$ and an action of P_2 on it. Therefore, if the constraint $\det\widehat{R} = 1$ is dropped, the total number of configurations grows factorially with n. The growth rate of the number of matrices giving $\det\widehat{R} = 0$ is factorial as well, because it is sufficient to equate any two rows or columns to vanish the determinant.

A convenient pictorial representation can be given in terms of oriented graphs on n vertices, $\mathcal{G}_n$. For it we interpret the B matrices as the adjacency matrices of the graphs. If $B_{ij} \neq 0$, we draw an arrow from the j'th to the i'th vertex. There are exactly two links going away from every vertex and exactly two arriving at it (one corresponding to P_1 and the other to P_2). The determinant of $\widehat{R}$ can be rewritten as the sum over all possible collections of oriented self-avoiding loops $\{L_1 \ldots L_p\}$ in the graph $\mathcal{G}_n$

$$\det(I-B) = \exp\Big(-\sum_{k=1}^{\infty} \frac{\mathrm{tr}\ B^k}{k}\Big) = 1 + \sum_{\{L_1\ldots L_p\}} (-1)^p \prod_{k=1}^{p} \prod_{(ji)\in L_k} B_{ij}\,, \tag{4.3}$$

where $\prod_{(ji)\in L_k}$ goes over links constituting a loop L_k.

If we interpret B as the incidence matrix (*i.e.,* if $\widehat{R}_{ij} \neq 0$, the i'th edge is incident to the j'th vertex), then corresponding graphs are always collections of disjoint circles. These give the cycle structure of $P_1^{-1}P_2$.

Obviously $\det\widehat{R} = 1$, if the orgraph $\mathcal{G}_n$ has no closed contours. It does not necessary mean that $\mathcal{G}_n$ is a tree, because the loops in Eq. (4.3) are all oriented, *i.e.,* one is allowed to travel only in directions shown by the arrows. If we had the stronger constraint, $\sum_{j=1}^{n} |B_{ij}| \leq 1$ (*i.e.,* exactly one arrow goes from every vertex), then the only possible values of $\det\widehat{R}$ would be 0, 1 and 2. In the case $\det\widehat{R} = 1$, trees only would contribute. This simple model gives a solution to the analogous counting problem for 2-dimensional spherical triangulations (cf. Ref. [7]).

Let us formulate the next intermediate result as

PROPOSITION 3. Let $|\widehat{\mathcal{M}}^p_n|$ be the number of the equivalence classes of matrices $\{M \in SL(n,Z) \mid M' \cong M''$, if $M' = \omega_1 M'' \omega_2$ $(\omega_{1,2} \in O(n,Z))\}$ obeying the additional constraints $(M^{\mathrm{T}}M)_{kk} = (MM^{\mathrm{T}})_{kk} = p$, $\forall k$. Then, as $n \to \infty$ and p finite, $|\widehat{\mathcal{M}}^p_n| \leq \exp(\lambda n)$ with some finite constant λ.

PROOF To prove that, we can use the following facts from the theory of group lattices (see, for example, Ref. [18]):
(i) $SL(n,Z)$ is a lattice in $SL(n,R)$, *i.e.*, $SL(n,R)/SL(n,Z)$ is connected and has a finite volume with respect to any Haar measure on $SL(n,R)$.
(ii) A fundamental domain of $SL(n,Z)$ in $SL(n,R)$ can be conveniently described in terms of the Iwasawa decomposition: $M = \omega\alpha\eta$, where $\omega \in SO(n)$; $\alpha = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$ (with all $\alpha_k > 0$ and $\prod_{i=1}^n \alpha_i = 1$); $\eta \in N$, where N is the group of upper-triangular unimodular matrices (*i.e.*, if $k > l$, $\eta_{kl} = 0$; on the diagonal $\eta_{kk} = 1$; and, for $k < l$, η_{kl} is arbitrary). A Siegel set in $SL(n,R)$ is a set of the form $S_{tu} = \Omega A_t N_u$ defined by the inequalities: $A_t = \{\alpha : \alpha_i \leq t\alpha_{i+1}\ (1 \leq i < n)\}$, $N_u = \{\eta : |\eta_{kl}| \leq u\ (1 \leq k < l \leq n)\}$ and Ω coincides with the total $SO(n)$ group. The fundamental domain is the Siegel set S_{tu} with $t = \frac{2}{\sqrt{3}}$ and $u = \frac{1}{2}$. The Haar measure in this parametrization is

$$d\mu = d\omega \prod_{i<j} \frac{\alpha_i}{\alpha_j} \prod_{i=1}^{n} d\alpha_i \prod_{i<j} d\eta_{ij}\,. \tag{4.4}$$

From the constraint

$$\alpha_k^2 + \sum_{i<k} \alpha_i^2 \eta_{ik}^2 = p \tag{4.5}$$

we find that $\alpha_k^2 \leq p$ for all k. Together with the unimodularity property of matrices, it gives

$$\prod_{j=2}^{n} (1 + \prod_{i<j} \eta_{ij}^2) \leq p^n\,. \tag{4.6}$$

From the observation that

$$(M^{\mathrm{T}}M)_{ij} = \alpha_i^2 \eta_{ij} + \sum_{k<i} \alpha_k^2 \eta_{ki}\eta_{kj} \qquad (i < j) \tag{4.7}$$

is a dilute matrix, it follows that η_{nm} is dilute as well: in any row of η there are only a finite number of non-zero elements. These equations mean that only a finite number of inequalities $|\eta_{kl}| \leq \frac{1}{2}$ can be violated for every value of l.

The action of $SL(n,Z)$ divides the $SL(n,R)$ group manifold into cells, each of them identical to the fundamental domain. The equation

$$\alpha_i = \frac{2}{\sqrt{3}}\alpha_{i+1} \tag{4.8}$$

defines a hypersurface separating two different cells.

Now, we can use a standard trick in analytic number theory: to estimate the number of distinct $SL(n,Z)$ matrices obeying some given inequalities, we can simply count the number of cells inside a corresponding region in the $SL(n,R)$ group manifold.

As we are interested in counting the equivalence classes under arbitrary permutations of rows and columns, let us assume in addition that

$$\alpha_i \geq \alpha_{i+1}\,. \tag{4.9}$$

For arbitrary $k \leq n$

$$\prod_{i=1}^{k} \alpha_i^2 = \det_{1\leq i,j\leq k} (M^{\mathrm{T}}M)_{ij} \tag{4.10}$$

and the permutations of rows rearrange the n-tuples changing values of α's, not simply permuting them. We can choose one in which the inequalities(4.9) are satisfied. It is specific to our particular case.

The unimodularity and the constraints (4.5) together with the inequalities (4.9) imply that, in the $n \to \infty$ limit, only a finite number of equations (4.8) can be simultaneously satisfied. If this number equals k, we can estimate the number of distinct configurations as $(cn)^k/k!$ (with some constant c). As upper-triangular unimodular matrices form a group over integers, we can always obey the inequalities $|\eta_{ij}| \leq 1/2$ $\forall(i,j)$. Analogously, for the left multiplications by $SO(n,Z)$ matrices, we can choose any coordinates on the coset $SO(n,R)/SO(n,Z)$. Then, as follows from the theory of group lattices, the integrality constraints on matrix elements can have only one or no solution for any fixed tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

The number of equivalence classes of dilute upper-triangular η_{ij} matrices is exponentially bounded. The combinatorial reason for that is that they have, in a general position, more symmetry than densely filled matrices from the N group. If $\eta_{nm} = 0$ for $k \leq n, m \leq l$, then $\sigma\eta\sigma^{-1} = \eta$ for all permutations σ acting only on the rows and columns from k to l and identical on the others ($\sigma \in I_{r'} \oplus S_q \oplus I_{r''}$, $q = l - k + 1$). Let us take one of such matrices and try to draw a consequence of this enhanced symmetry. We have

$$\sigma(\omega\alpha\eta)\sigma^{-1} = \omega'\alpha'\eta'\,. \tag{4.11}$$

As $\omega' = \sigma\omega\sigma^{-1} \in SO(n)$ and $\eta' = \sigma\eta\sigma^{-1} \in N$, $\alpha' = \sigma\alpha\sigma^{-1}$ has to represent an integer point in the α space (*i.e.,* corresponding to a matrix from $SL(n,Z)$). Let $\ell_{\min}$ be a minimal distance between any integer points. If α_i is permuted with α_j and $|\log(\alpha_i/\alpha_j)| << \ell_{\min}$, then, in a general position inside a lattice cell, there must be $\alpha_i = \alpha_j$, because otherwise there would be 2 different integer points in the fundamental domain. Thus, we conclude that, in a general position, the product $\alpha\eta$ possesses the same symmetry under permutations as the dilute η matrix alone. Then, in the $n \to \infty$ limit, the number of equivalence classes of such products has to be of the same order as for the η matrices. $\Box$

From the pure mathematician's point of view, the argumentation given above is not satisfactory. Hopefully, a mathematically minded reader will be able to complete the proof (or, maybe, find a loophole in it). Let us accept the claim of this proposition as a plausible hypothesis. Then we can conclude that

THEOREM 1 The number of 3-dimensional simplicial manifolds constructed of n tetrahedra and having the trivial homology group, $H_1(C, Z) = 0$, grows at most exponentially with n: $|C_n| < e^{\lambda n}$, for some finite constant λ.

PROOF To finish the proof, we have to show that, for any matrix $M \in \widehat{\mathcal{M}}_n^3$ from a given equivalence class, there are exponentially many simplicial complexes accociated with it. In terms of abelian presentations, the problem is reduced to estimating the number of inequivalent multiplications by upper triangular $N_1 \times (N_3 + 1)$ matrices. A multiplication by such a matrix from the left corresponds to a change of the basis of relators in a π_1 presentation generated by AC2 and AC4 moves only. Let us remember that we are considering simplicial complexes. These changes of bases corresponds to formal 2-deformations of 2-spines of complexes. Therefore their number is restricted from above by the number of corresponding spanning trees in $\widetilde{K_1} \subset \widetilde{C}$. As the valency of the trees is equal to 4, the statement follows. □

THEOREM 2 The number of 3-dimensional simplicial manifolds constructed of n tetrahedra and having a fixed homology group $H_1(C, Z)$ grows at most exponentially with n: $|C_n| < e^{\lambda n}$, for some finite constant λ.

PROOF As was pointed out in Section 2, an arbitrary presentation matrix can be transformed into a unique diagonal form by pre- and post-multiplications by $SL(n, Z)$ matrices. Such multiplications correspond to changes of basis of relators as well as generators. Therefore, the statement is a consequence of the previous theorem. □

It is natural to assume that, having fixed the Betti number b_1 only, one gets a factorial growth of the number of complexes as a function of a volume. Indeed, we can see that the number of inequivalent presentation matrices grows faster than an exponential in this case. On the other hand, it is a piece of mathematical folklore that there is always at least one 3-manifold associated with a given abelian presentation[5]. Although, such a manifold should not necessarily be simplicial, it is plausible assumption that the simpliciality condition is not restrictive enough to cut down the growth rate.

5. Reduced model and connection with the 2-dimensional loop gas model

As has been already discussed, in 3 dimensions there are several natural choices of classes of complexes to define the partition function over. And the question of universality classes is open. The simplest choice could be the set of 3-spheres with imbedded 2-collapsible spines. It looks very natural from the viewpoint of Section 3. We restrict acceptable presentations to the set of obviously trivial ones. Namely, to those that can

[5] We are indebted to V.Turaev for communicating this fact to us.

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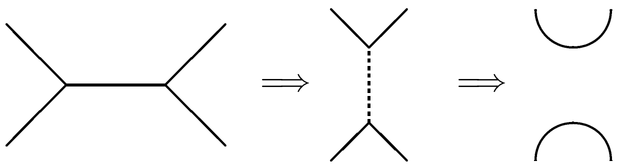
be reduced to the empty one, $\langle | \rangle$, by using only AC2 and AC4 moves as follows from the little Wright's theorem. So, we arrive at the partition function for the reduced model (cf. Eq. (3.3) and (3.4))

$$\mathbf{Z}^{(2)}_{\rm red}(\alpha,\mu) = \sum_{\{C\cong S^3\}} \sum_{\{K_2\subset C | K_2 \overset{2}{\nearrow\!\!\searrow} *\}} \sum_{\{T\subset K_2\}} e^{\alpha N_0 - \mu N_3} = \\ = \sum_{N_3, N_0} \sum_{\{P_{N_3+1;N_3+N_0}\cong \langle | \rangle\}} e^{\alpha N_0 - \mu N_3}\,, \tag{5.1}$$

where $\sum_{\{C\cong S^3\}}$ is the sum over all simplicial 3-spheres; $\sum_{\{K_2\subset C | K_2 \overset{2}{\nearrow\!\!\searrow} *\}}$ is the sum over 2-collapsible spines, K_2, embedded in C; $\sum_{\{T\subset K_2\}}$ is the sum over all spanning trees in K_2. The abelianizations of all the presentations from the sum $\sum_{\{P_{N_3+1;N_3+N_0}\cong \langle | \rangle\}}$ have upper-triangular unimodular matrices.

Let us remind a reader that, by definition, a spine is a 2-skeleton K_2 of a complex C in a decomposition with only one 3-cell. In the simplicial context, this 3-cell is a ball B_3 with a triangulated boundary. K_2 is obtained by some pair-wise identification (gluings) of all boundary triangles. If K_2 is 2-collapsible, one can use recursively only one gluing operation: the identification of 2 triangles sharing a common link on the boundary, in other words, a folding along the link. This can simulate any sequence of elementary 2-collapses and expansions, with an elementary 2-ball being just a single triangle.

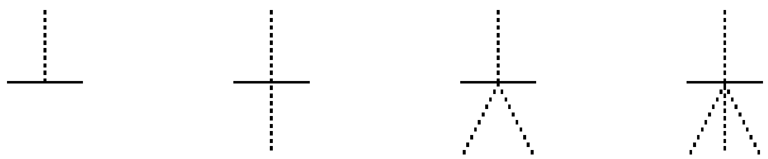
In the dual language, one starts with an arbitrary spherical 3-valent fat graph and applies the move which can be represented as the flip of a link with the subsequent elimination of it:



(5.2)

Both of these moves are well known and have been used in Monte-Carlo simulations of triangulated surfaces [7]. Having glued all triangles, one finishes with a collection of self-avoiding closed loops, the number of which equals $N_0 - 1$. We weight every loop with the numerical factor e^α.

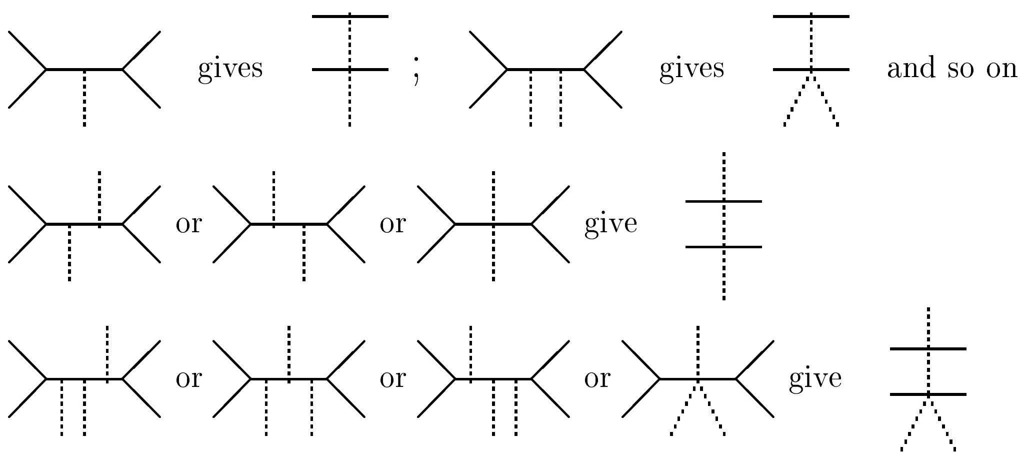
It is convenient, instead of erasing a links after a flip, to decorate it with the dashed propagator and keep its track in a process of subsequent foldings. To represent such configurations, we need to introduce the infinite set of vertices



and so on,

(5.3)

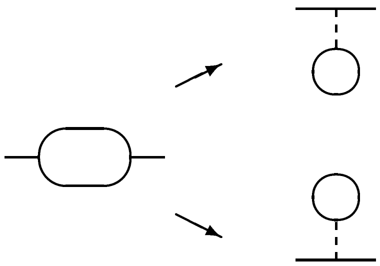
which are generated by flips. For example,



and so on.

In the end, we obtain planar diagrams with two types of propagators: solid ones produce closed loops while dashed form arbitrary clusters. The total number of vertices is equal to $2(N_3 + 1)$.

Different sequenses of flips can lead to differently looking planar diagrams corresponding to the same 3-dimensional complex. The simplest example is given by the two possible self-energy reductions:



(5.4)

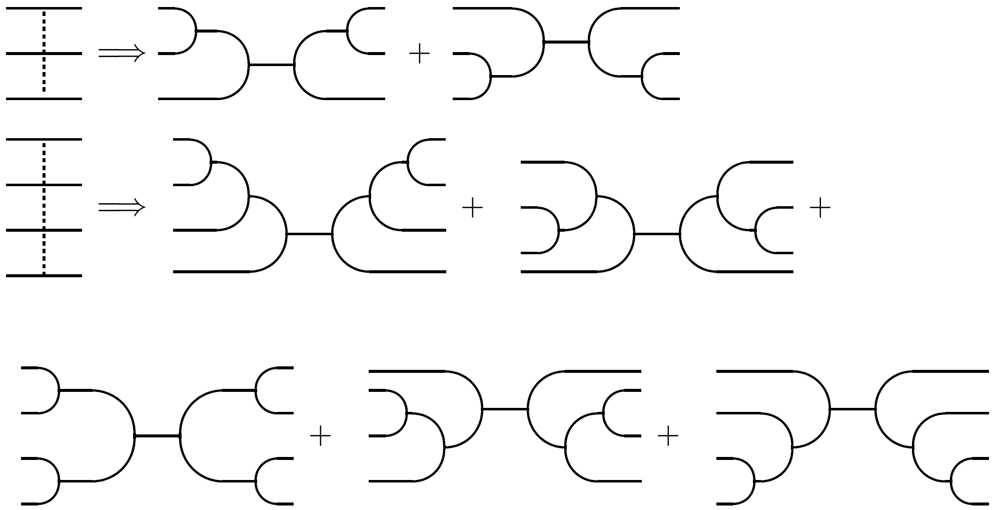
It is easy to see that this overcounting exactly corresponds to taking a sum over all spanning trees $\{T \subset K_2\}$ in Eq. (5.1). More precisely, links chosen to fold along are not in T.

Thus, the model can be represented as a non-local gas of self-avoiding closed loops with the partition function

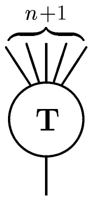
$$Z^{(2)}_{\rm red}(\nu, \mu) = \sum_{\{\mathcal{D}\}} w_{\mathcal{D}}\, e^{\alpha N_0 - \mu N_3}\,, \tag{5.5}$$

where $\sum_{\{\mathcal{D}\}}$ is the sum over all the diagrams. The weight $w_{\mathcal{D}}$ is equal to the number of inequivalent starting configurations giving the same planar diagram $\mathcal{D}$ (can be 0). The number of closed loops is simply equal to $N_0 - 1$ and α can be identified with the inverse Newton constant.

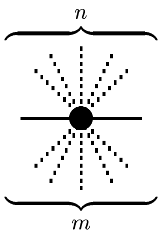
Let us take an arbitrary planar graph with dashed propagators. In order to obtain all possible starting configurations, we have to flip back dashed links in all possible ways. For any given dashed cluster, a result is a subset of all planar 3-valent graphs. Hence, the number of them is exponentially bounded as a function of the number of links. Let us give a few illustrative examples. First, flips have to be performed in linear clusters giving all possible planar 3-valent trees doubled in a “mirror”:



and so on. The number of initial configurations for a linear claster of n dashed links, C_n, is given by Catalan's numbers generated by the function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n := \sum_{n=0}^{\infty}$$  $$x^n = \frac{1}{2x}(1-\sqrt{1-4x})\,. \tag{5.6}$$

After that dashed propagators attached to the ends of the linear cluster have to be reattached in all possible ways to links of these “doubled trees”. The number of configurations can be estimated as follows. First, we have to calculate the generating function for the number of possible attachments of dashed propagators to a solid line:

$$q(\nu,\mu) := \sum_{n,m}$$  $$\nu^n \mu^m\,. \tag{5.7}$$

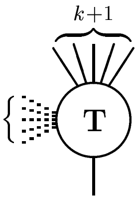
It can be obtained from the equation

$$q(\nu,\mu) = \Big[\frac{\nu}{1-\mu} + \frac{\mu}{1-\nu} - \nu\mu\Big] q(\nu,\mu) + 1\,, \tag{5.8}$$

which gives

$$q(\nu,\mu) = \frac{(1-\nu)(1-\mu)}{(1-\nu-\mu)^2 - (\nu+\mu)\nu\mu + \nu^2\mu^2}\,. \tag{5.9}$$

Second, we have to know the generating function for the number of all possible attachments of dashed propagators to a k-legged tree on the left (or on the right but not on both sides):

$$t(x,\nu) := \sum_{k,n} n\Big\{$$


$$x^k\nu^n\,; \qquad t(x,0) = C(x)\,. \tag{5.10}$$

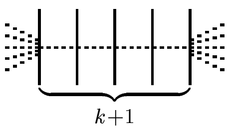
The equation for it is simply

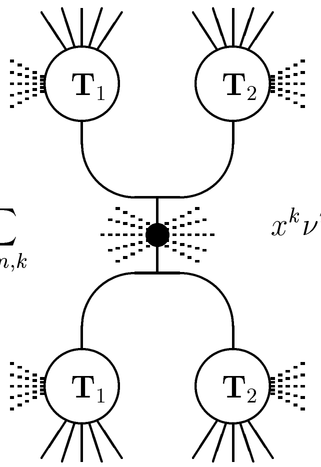
$$t(x,\nu) = xC(x)\frac{t(x,\nu)}{1-\nu} + 1\,, \tag{5.11}$$

.which gives

$$t(x,\nu) = \frac{1-\nu}{1-\nu-xC(x)}\,. \tag{5.12}$$

Then, the quantity we need to calculate is

$$T(x,\nu,\mu) = \sum_{n,m,k} n\Big\{$$


$$\Big\}_m\, x^k\nu^n\mu^m = \sum_{n,m,k}$$


$$x^k\nu^n\mu^m$$

$$= q(\nu,\mu)\left\{1 + \frac{x(1-\nu)^2(1-\mu)^2}{\Big((1-\nu)^2 - xC(x)\Big)\Big((1-\mu)^2 - xC(x)\Big)}\right\}. \tag{5.13}$$

Newly appearing clusters have to be expanded in all possible ways according to the same procedure. What is important is that their number is restricted by the coefficients of $q(\nu,\mu)$, which is a factor in (5.13), and does not depend on a volume of the trees. Therefore, one simply iterates till all dashed links disappear. The combinatorial reason for the existence of an exponential bound for the number of obtained configurations is clear: they are all planar!

We have just seen that the weights $w_{\mathcal{D}}$ in Eq. (5.5) are recursively calculable. However, it is difficult to really take account of the entropy of configurations and the model (5.5) seems to be analyticly unsolvable. Therefore, as was first proposed in Ref. [6], let us consider the localized version of it, namely, the dense phase of the self-avoiding loop gas matrix model [8] (in the sequel simply loop gas (LG) model):

$$\begin{aligned} Z_{LG} &= \int d^{N^2}Y d^{N^2}\prod_{\nu=1}^{n} X_\nu \exp\Big[-\frac{N}{2}\mathrm{tr}\,Y^2 - \frac{N}{2}\sum_{\nu=1}^{n}\mathrm{tr}\,X_\nu^2 + \frac{\lambda N}{2}\sum_{\nu=1}^{n}\mathrm{tr}\,YX_\nu^2\Big] \\ &= \int d^{N^2}Y \exp\Big[-\frac{N}{2}\mathrm{tr}\,Y^2 - \frac{n}{2}\mathrm{tr}^{\,2}\log\Big(1\otimes 1 - \lambda(Y\otimes 1 + 1\otimes Y)\Big)\Big]\,. \end{aligned} \tag{5.14}$$

Here Y and X_ν are hermitian $N \times N$ matrices. We have attached the lower index to the gaussian X_ν variable to weight every closed loop with the factor $n = e^\alpha$. $\lambda = e^{-\mu/2}$. In the $N \to \infty$ limit, this model generates planar diagrams having only the simplest 3-valent vertices from the whole series (5.3). For such diagramms, all the weights in Eq. (5.5) are trivial: $w_{\mathcal{D}} = 1$.

Let us suppose that this truncated model bears some qualitative features of 3-dimensional simplicial gravity. It seems to be a reasonable assumption, because, as follows from the results of Ref. [8], the corresponding universality class is very large and includes all local perturbations of the model (5.14). On the other hand, multiple overlapping sequences of the flips can be regarded as a kind of renormalization group procedure, which could draw any given system to a stable fixed point in some imaginable space of all planar-graph models. It looks a bit more natural in dual terms. One takes an arbitrary spherical ball with a huge triangulated boundary and uses subsequently and randomly the folding operation (dual to the flip). Soon the boundary triangulation will be randomized. If this randomization has appropriate statistical properties, we could expect to find many models falling in the same universality class and, therefore, having the same continuum limit.

The critical behavior of the loop gas matrix model is well known. For $n < 2$, it describes 2-d gravity interacting with $c < 1$ conformal matter

$$c = 1 - 6\frac{(1-g_0)^2}{g_0}\,; \qquad n = -2\cos\pi g_0\,, \tag{5.15}$$

while for $n > 2$ the corresponding matter is non-critical and the model trivializes. In this phase, the number of closed loops is proportional to the volume and the mean length of each remains finite.

In the vicinity of a critical point μ_c, the partition function behaves as

$$Z_{LG} \approx (\mu_c - \mu)^{2-\gamma_{\rm str}}\,. \tag{5.16}$$

This formula defines the famous string susceptibility exponent $\gamma_{\rm str}$.

Two main quantities of interest are the mean lattice volume

$$\langle N_3\rangle_{LG} = \frac{\partial}{\partial\mu}\log Z_{LG} \approx -(2-\gamma_{\rm str})\frac{\mu_c}{\mu_c - \mu} \tag{5.17}$$

and the mean lattice curvature

$$\begin{aligned}\langle N_0\rangle_{LG} &= \frac{\partial}{\partial\alpha}\log Z_{LG} \approx (2-\gamma_{\rm str})\frac{n\mu_c'(n)}{\mu_c(n)-\mu} - \frac{\partial\gamma_{\rm str}}{\partial\alpha}\log|\mu_c-\mu| \\ &\approx -\frac{\mu_c'}{\mu_c}\langle N_3\rangle_{LG} + \frac{\partial\gamma_{\rm str}}{\partial\alpha}\log\langle N_3\rangle_{LG}\,,\end{aligned} \tag{5.18}$$

where the coefficients can be calculated explicitly using the Gaudin and Kostov's exact solution [8]

$$\frac{\partial}{\partial n}\log\mu_c = -\frac{1}{2}\frac{1}{n+2}\,; \qquad \frac{\partial\gamma_{\rm str}}{\partial n} = \frac{1}{2\pi g_0^2\sin\pi(1-g_0)}\,. \tag{5.19}$$

If $2 - n \ll 1$, then [6]

$$\langle N_0 \rangle_{LG} \approx \frac{1}{4} \langle N_3 \rangle_{LG} + \frac{1}{\pi\sqrt{2-n}} \log \langle N_3 \rangle_{LG} \,. \tag{5.20}$$

If $n > 2$, then γ_{str} is independent of n and $\langle N_0 \rangle_{LG}$ is proportional to $\langle N_3 \rangle_{LG}$. As was shown in Ref. [19], in this region of n, $\gamma_{\mathrm{str}} = +\frac{1}{2}$ in the dense phase of the loop gas model.

This behavior is strikingly similar to the results of numerical simulations. In Ref. [4], a phase transition in the model (1.2) with respect to the Newton coupling, α, was found. In the "elongated" phase, $\alpha > \alpha_c$, the mean number of vertices, $\langle N_0 \rangle$, is strictly proportional to a volume, $\langle N_3 \rangle$. In the "crumpled" phase, $\alpha < \alpha_c$, $\langle N_0 \rangle$ is a non-trivial function of $\langle N_3 \rangle$. The analogy with the loop gas matrix model suggests that the most probable scaling is $\langle N_0 \rangle \approx c_1 \langle N_3 \rangle + c_2 \log \langle N_3 \rangle$ with c_2 singular at the critical point α_c. Presumably, it is a type of this singularity that can be, in principle, calculated in continuum theory in order to compare predictions of both approaches. The obvious problem for such a hypothetical comparison is that α is a bare coupling. Therefore, in the lattice model, only critical points with respect to it could show some universal features. In any case, Eq. (5.20) looks reasonable. For the dimensionful mean curvature, $\langle R \rangle = 2\pi a \langle N_0 \rangle_{LG} + ca \langle N_3 \rangle_{LG}$, as a function of the voulume, $V = a^3 \langle N_3 \rangle_{LG}$, we find

$$\langle R \rangle \approx \frac{c_1}{a^2} V + \frac{2a}{\sqrt{2-n}} \log V + \dots \tag{5.21}$$

As the scalar curvature has to undergo an additive renormalization, it is natural to find a linear volume term singular in the continuum limit: $a \to 0$, V finite. The logarithmic in volume term can have in this limit a finite coefficient, if simultaneously $n \to 2 - 0(a^2)$. This is quite physical behavior.

Of course, the loop gas matrix model cannot give us precise quantitative information about simplicial gravity. However, it is very plausible that it has qualitatively the same phase structure as models (1.2) and (3.3) and may be quite instructive from this point of view.

6. Discussion

1. Combinatorial group theory gives a natural mathematical framework and sets up a standard language for physical problems related to lattice models of 3-dimensional quantum gravity. All the formal group constructions with relators and generators have a natural geometrical realization in terms of 2-dimensional complexes (or fake surfaces, in a less formal parlance). And vise versa, geometrical constructions can be formalized in the group theory terms. It would be interesting to find physical models which could be formulated and solved entirely in terms of abelian presentations. It might be a mathematically adequate way to make physically meaninful approximations.

2. To determine a growth rate for different classes of complexes is the first necessary step in the process of constructing and investigating simplicial gravity models. It is an interesting field of research in its own right, altough more mathematical in spirit. Explicit forms of the asymptotic expansions would give rigorous and exact solutions of the correponding physical models.

3. The status of the reduced model (5.1) is not yet absolutely clear. As it was formulated in Eq. (5.1), it looks as a genuine simplicial gravity model with some peculiar matter coupled to it. It is not clear if this matter has any local description in terms of fields in a manifold. The first natural question about the model to ask is if all simplicial spheres are really counted in $\mathbf{Z}^{(2)}_{\mathrm{red}}(\alpha, \mu)$. Or, in other words, whether any simplicial 3-sphere allows for embedding at least one 2-collapsible spine in it. At first thought, Zeeman's Dunce Hat could serve as a source of counterexamples. However, triangles lying inside an initial ball can form such a fake surface. Even in the simplest case of 2 tetrahedra a Zeeman's Dunce Hat can easily be obtained. And, say, the complex constructed in Ref. [20] cannot be a counterexample because it is not simplicial. It seems to be an uneasy question in general. Say, the similar statement in case of simply-connected manifolds would automaticly prove the famous Poincaré hypothesis.

4. The model (5.1) can be successfully simulated numerically with help of the Monte Carlo technique. Results of the simulations as well as an account of the algorithm will be reported elsewhere [21]. Let us simply mention here that they support the hypothesis that both the reduced 3-dimensional model (5.1) and the 2-dimensional loop gas model belong to the same universality class in the continuum limit, with the Newton coupling playing a role analogous to the central charge in 2D gravity. It would be extremely interesting if the original model (3.3) had a similar continuum limit. In principle, it would not contradict to what is known from numerical simulations of the pure 3D gravity model. If we remember that the analogous 2-dimensional matrix model discribes the $c = -2$ matter coupled to gravity, which is topological, then it might seem less surprising that our 3-dimensional construction is equivalent to some 2-dimensional model.

Acknowledgments

I would like to thank J.Ambjørn, B.Durhuus, V.Kazakov, N.Kawamoto, V.Turaev and Y.Watabiki for the fruitful discussions. This work has been partially supported by the grants RFFI-97-02-17927 and INTAS-96-0524.

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